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ON THE AUTOMORPHISM GROUPS OF COUNTABLE BOOLEAN ALGEBRAS

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ABSTRACT
We prove categoricity results for the class $\{\langle B, \mathrm{Aut}(B)\rangle \mid B$ is a countable BA$\}$.

§ 0. Introduction

If A is a Boolean algebra (hereafter denoted by BA) let $\mathrm{At}(A)$ denote the set of atoms of A. Let $K_0' = \{A \mid A$ is a countable or finite BA and $|\mathrm{At}(A)| \neq 1\}$. A and B always denote members of K_0' and $\mathrm{Aut}(A)$ denotes the automorphism group of A.

J. D. Monk conjectured that for every $A, B \in K_0'$, $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$ implies $A \cong B$. In [4] he proved the special case of this conjecture when $\mathrm{At}(A)$ is finite. R. McKenzie [3] and independently S. Shelah refuted this conjecture by showing two non-isomorphic BA's in K_0' having isomorphic automorphism groups. On the other hand McKenzie in [3] proved that Monk's conjecture is true when A is atomic or even when A has a maximal atomic element. The question what information about A is contained in $\mathrm{Aut}(A)$ when A is an arbitrary member of K_0' remained unsettled. Thus, for instance, McKenzie in [3] asked whether $\mathrm{Aut}(A) \equiv \mathrm{Aut}(B)$ implies $A \equiv B$.

By interpreting in $\mathrm{Aut}(B)$ a certain fragment of the second order logic of B (Theorem 2.15(d)) we answer McKenzie's question affirmatively. We actually prove a somewhat stronger result (Corollary 2.17):

THEOREM. *For every complete theory of BA's T there is a sentence φ_T in the language of groups such that for every $B \in K_0'$ $B \models T$ iff* $\mathrm{Aut}(B) \models \varphi_T$.

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Another corollary from 2.15(d) is, that if B is infinite and atomic then B^{II} is explicitly interpretable in $\mathrm{Aut}(B)$ (see § 1), where B^{II} is the second order model of B, i.e.

$$B^{\mathrm{II}} = \langle B, \mathcal{R}_1(B), \cdots, \mathcal{R}_n(B), \cdots; \cup, \cap, -, 0, 1, E_1, \cdots, E_n, \cdots \rangle,$$

$\mathcal{R}_n(B)$ is the set of n-place relations on B and

$$\langle a_1, \cdots, a_n, r \rangle \in E_n \quad \text{iff} \quad r \in \mathcal{R}_n(B) \quad \text{and} \quad \langle a_1, \cdots, a_n \rangle \in r.$$

Using a theorem from [2] we conclude:

THEOREM. ($V = L$) *If* A *has a maximal atomic element and* $\mathrm{Aut}(B) \equiv \mathrm{Aut}(A)$ *then* $B \cong A$.

Thus assuming $V = L$ McKenzie's result can be strengthened.

By [2] it is however consistent that there are $2^{\aleph_0}$ non-isomorphic countable superatomic BA's with elementarily equivalent automorphism groups.

In §3 we prove the following theorem: *assume* $V = L$ *then if* $\mathrm{Aut}(A) \equiv \mathrm{Aut}(B)$ *then* $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$.

We do not know the answer to the following question: Let B be a countable BA; is it possible to interpret in $\mathrm{Aut}(B)$ the set of all isomorphism types of countable BA's whose automorphism group is isomorphic to $\mathrm{Aut}(B)$? To make the question precise let $M(B) = \langle \mathcal{T}(B), U : \cup, \cap \rangle$, where $\mathcal{T}(B)$ is the set of isomorphism types of countable BA's whose automorphism group is isomorphic to $\mathrm{Aut}(B)$, $\cup, \cap \subseteq \mathcal{T}(B) \times U^3$ and for every $\tau \in \mathcal{T}(B)$ the relations

$$\bigcup_{\tau} = \{\langle x, y, z \rangle \mid \langle \tau, x, y, z \rangle \in \cup\}$$

and

$$\bigcap_{\tau} = \{\langle x, y, z \rangle \mid \langle \tau, x, y, z \rangle \in \cap\}$$

define a BA on their (common) domain of isomorphism type τ.

Let $K = \{\mathrm{Aut}(B) \mid |B| \leqq \aleph_0\}$ and $L = \{M(B) \mid |B| \leqq \aleph_0\}$; is L explicitly interpretable in K?

We conjecture that the answer to this question is negative, even if we restrict ourselves to the class of all models of some complete theory of BA's T.

See [6] and [7] for results complementary to the results in this paper.

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§ 1. Definitions and notations

DEFINITION. Let K and K^* be classes of models in the language L and L^* respectively, $h : K \to K^*$ be an onto function. Then K^* is explicitly interpretable in K relative to h, if there are first order formulas of L $\varphi_U(x_1,\cdots,x_n)$, $\varphi_{Eq}(x_1,\cdots,x_n,y_1,\cdots,y_n)$, for every m-place relation symbol $R \in L^*$ $\varphi_R(x_1^1,\cdots,x_n^1,\cdots,x_1^m,\cdots,x_n^m)$ and for every m-place function symbol $F \in L^*$ $\varphi_F(x_1^1,\cdots,x_n^1,\cdots,x_1^m,\cdots,x_n^m,y_1,\cdots,y_n)$, such that for every $M \in K$ if $M^* = h(M)$ then there is an onto function $f:\{\langle a_1,\cdots,a_n\rangle \mid M \models \varphi_U[a_1,\cdots,a_n]\} \to |M^*|$ such that for every $\vec{a},\vec{b} \in \mathrm{Dom}(f)$ $f(\vec{a}) = f(\vec{b})$ iff $M \models \varphi_{Eq}[\vec{a},\vec{b}]$, for every $R \in L^*$ and $\vec{a}^1,\cdots,\vec{a}^m \in \mathrm{Dom}(f)\langle f(\vec{a}^1),\cdots,f(\vec{a}^m)\rangle \in R^{M^*}$ iff $M \models \varphi_R(\vec{a}^1,\cdots,\vec{a}^m)$ and for every $F \in L^*$ $\vec{a}^1,\cdots,\vec{a}^m,\vec{b} \in \mathrm{Dom}(f)$ $f^{M^*}(f(\vec{a}^1),\cdots,f(\vec{a}^m)) = f(\vec{b})$ iff $M \models \varphi_F(\vec{a}^1,\cdots,\vec{a}^m,\vec{b})$.

The constants and operations in a BA are denoted by $0, 1, \cup, \cap, -$, and the partial order induced on the BA is denoted by $\subseteq$. If B is a BA and $C \subseteq B$, then $\mathrm{cl}(C)$ denotes the subalgebra of B generated by C, and $\mathrm{Alg}(C)$ denotes its algebraic closure, that is, $\mathrm{Alg}(C) = \{b \mid b \in B$ and there is a formula $\varphi(x,\bar{y})$ and $c_1,\cdots,c_n \in C$ such that $B \models \varphi[b,c_1,\cdots,c_n]$, and there are just finitely many elements b' of B such that $B \models \varphi[b',c_1,\cdots,c_n]\}$. $\mathrm{At}(B)$, $\mathrm{As}(B)$, $\mathrm{Al}(B)$ denote the set of atoms of B, the set of atomic elements of B and the set of non-zero, non-maximal atomless elements of B respectively. Barred letters $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ denote subsets of $\mathrm{At}(B) \cup \mathrm{Al}(B)$. If $a \in B$ we define $\bar{a} = \{b \mid b \subseteq a$ and $b \in \mathrm{At}(B) \cup \mathrm{Al}(B)\}$; the statement $\bar{b} \subseteq \mathrm{At}(B) \cup \mathrm{Al}(B)$ does not imply that for some $c \in B$ $\bar{c} = b$. The direct product of the BA's A and B is denoted by $A \times B$.

A function $f:|M| \to |N|$ is called elementary, if for every $a_1,\cdots,a_n \in \mathrm{Dom}(f)$ and for every $\varphi(x_1,\cdots,x_n)$ in the language of $M : M \models \varphi[a_1,\cdots,a_n]$ iff $N \models \varphi[f(a_1),\cdots,f(a_n)]$.

Id denotes the identity function. If $\mathrm{Dom}(f) \subseteq A$, $\mathrm{Rng}(F) \subseteq B$ and f is elementary then $\mathrm{cl}(f)$ denotes the unique extension of f to an elementary function whose domain is $\mathrm{cl}(\mathrm{Dom}(f))$.

The universe of the model M is denoted by $|M|$; $b, \bar{a}, f$ are used to denote elements of the model as well as variables in the language, but in "$\varphi(\bar{a})$" $\bar{a}$ denotes an individual variable, whereas in "$M \models \varphi[\bar{a}]$" $\bar{a}$ denotes an element of $|M|$ which satisfies φ. B denotes both a BA and its universe, the same convention is used for the group $\mathrm{Aut}(B)$. $\cong$ and $\equiv$ denote isomorphism and elementary equivalence respectively.

$M = \langle U_1,\cdots,U_n,\cdots;\ R_1^M,\cdots,R_n^M,\cdots,F_1^M,\cdots,F_n^M,\cdots\rangle$ denotes the model whose universe is $\bigcup_i U_i$ (the U_i's are not necessarily disjoint), and in addition to the relation and function symbols $\cdots R_i, \cdots, \cdots F_i \cdots$ the language of M contains for every U_n a unary predicate P_n such that $P_n^M = U_n$. We omit to mention those relations and functions which are natural, thus for instance $\langle \mathrm{Aut}(B), \mathrm{At}(B); \circ, \mathrm{Op} \rangle$ is abbreviated by $\langle \mathrm{Aut}(B), \mathrm{At}(B); \cdots \rangle$ where $\circ$ is the composition operation in $\mathrm{Aut}(B)$ and $\langle f, a, b \rangle \in \mathrm{Op}$ iff $f \in \mathrm{Aut}(B)$, $a, b \in \mathrm{At}(B)$ and $f(a) = b$.

If M and M' are models then $N = \langle M, M'; \cdots R_n \cdots \rangle$ is the model whose universe is $|M| \cup |M'|$, the relations and functions of N are those of M, M', the R_i's and unary prediates for $|M|$ and $|M'|$. (We assume the languages of M and M' are disjoint.)

The cardinality of C is denoted by $|C|$. $S_\omega(A)$ is the set of finite subsets of A.

§2. Let $K_1' = \{\mathrm{Aut}(B) \mid B \in K_0'\}$, $M_2(B) = \langle \mathrm{Aut}(B), \mathrm{At}(B), \mathrm{Al}(B); \cdots \rangle$ and $K_2' = \{M_2(B) \mid B \in K_0'\}$.

THEOREM 2.1. *K_2' is explicitly interpretable in K_1'.*

PROOF. This is a special case of corollary 1.5 in [5].

Let $\bar{B}$ be the completion of B, and $a_0 \in \bar{B}$ be the union of all atoms of B. (We regard B as a subalgebra of $\bar{B}$). Let $B^{\mathrm{AT}} = \langle \{a \cap a_0 \mid a \in B\} : \bigcup^B, \bigcap^B, -^{B^{\mathrm{AT}}}, 0, a_0 \rangle$ where $-^{B^{\mathrm{AT}}} a = a_0 - a$ and $B^{\mathrm{AL}} = \langle \{a - a_0 \mid a \in B\} : \bigcup^B, \bigcap^B, -^{B^{\mathrm{AL}}}, 0, -a_0 \rangle$ where $-^{B^{\mathrm{AL}}} a = -a_0 - a$, and let $B^{\mathrm{TL}} = B^{\mathrm{AT}} \times B^{\mathrm{AL}}$. We regard B as a subalgebra of B^{TL} and B^{AT} and B^{AL} as subsets of B^{TL}. Every $f \in \mathrm{Aut}(B)$ can be uniquely extended to an element of $\mathrm{Aut}(B^{\mathrm{TL}})$ and the set of such extensions is a subgroup of $\mathrm{Aut}(B^{\mathrm{TL}})$ so we regard $\mathrm{Aut}(B)$ as a subgroup of $\mathrm{Aut}(B^{\mathrm{TL}})$.

We now proceed to interpret $\langle \mathrm{Aut}(B), B^{\mathrm{AT}}; \mathrm{Op} \rangle$ in $\mathrm{Aut}(B)$ where $\langle f, a, b \rangle \in \mathrm{Op}$ iff $f \in \mathrm{Aut}(B)$, $a \in B^{\mathrm{AT}}$ and $b = f(a)$.

If f is an elementary function, let $\mathrm{Inv}(f) = \{a \mid a = f(a)\}$ and $\mathrm{fix}(f) = \{a \mid a \in \mathrm{At}(B) \cup \mathrm{Al}(B)$ and $\bar{a} \subseteq \mathrm{Inv}(f)\}$.

DEFINITION. Let $f : B \to B$, f will be called a good function if it is elementary, $\mathrm{Al}(B) \subseteq \mathrm{Dom}(f)$, for every $b \in \mathrm{Al}(B)$ $f(b) = b$ and for every $b \in \mathrm{Dom}(f)$ $(b - f(b)) \cup (f(b) - b)$ is the union of finitely many atoms.

Notice that if f is a good automorphism of B^{AT}, then $f \cup \mathrm{Id} \restriction B^{\mathrm{AL}}$ can be uniquely extended to an automorphism of B.

LEMMA 2.2. (a) *Let $f : B \to B$ be good, $C \subseteq B$ is finite and* $\mathrm{Dom}(f) \subseteq \mathrm{cl}(\mathrm{Al}(B) \cup C)$, *then f can be extended to a good automorphism of B.*

(b) *There is a good automorphism f of B such that* $\mathrm{Inv}(f) \subseteq \mathrm{cl}(\mathrm{Al}(B) \cup \{a_0\})$ *where a_0 is the maximal atomic element.*

(c) *If $C \subseteq B$ is finite, then there is a good $f \in \mathrm{Aut}(B)$ such that* $\mathrm{Inv}(f) = \mathrm{cl}(\mathrm{Al}(B) \cup C \cup \{a_0\})$.

(d) *If $b \in B$ and $|\mathrm{At}(B) - \bar{b}| \neq 1$ then there is a good $f \in \mathrm{Aut}(B)$ such that* $\mathrm{fix}(f) \cap \mathrm{At}(B) = \bar{b} \cap \mathrm{At}(B)$.

(e) *If $\bar{a} \subseteq \mathrm{At}(B)$ and $|\mathrm{At}(B) - \bar{a}| \neq 1$ then there is a good $f \in \mathrm{Aut}(B)$ such that* $\mathrm{fix}(f) \cap \mathrm{Aut}(B) = \bar{a}$.

Proof. (c) and (d) are corollaries of (b), and (e) is a corollary of (d); so let us prove (b).

By the remark preceding this lemma, it is sufficient to construct a good automorphism f of B^{AT}, such that $\mathrm{Inv}(f) = \{0,1\}$. So let us assume that $B = B^{\mathrm{AT}}$. If B is finite then any cyclic permutation of $\mathrm{At}(B)$ induces the desired automorphism; we thus assume that B is infinite.

Let $\{e_i \mid i < \omega\}$ be an enumeration of B. We define by induction on $n < \omega$ functions f_n; the induction hypothesis is:

$(*)$ f_n is good, $\mathrm{Dom}(f_n) = \mathrm{Alg}(\mathrm{Dom}(f_n))$ is finite, and $\mathrm{Inv}(f_n) = \{0,1\}$.

Let $f_0 = \{\langle 0,0\rangle, \langle 1,1\rangle\}$; since B is infinite f_0 satisfies $(*)$.

Let us now prove that if g satisfies $(*)$ and $x \in \mathrm{At}(B)$, then there is $g' \supseteq g$, such that $\mathrm{Dom}(g') = \mathrm{cl}(\mathrm{Dom}(g) \cup \{x\})$, and g' satisfies $(*)$. Let $a_0 \in \mathrm{At}(\mathrm{Dom}(g))$ and $x \subseteq a_0$. If $\overline{a_0}$ is finite, then $\overline{a_0} \subseteq \mathrm{Dom}(g)$, so define $g' = g$. Otherwise, let $y \in \overline{a_0} \cap g(\overline{a_0}) - \{x\}$, then $g \cup \{\langle x, y\rangle\}$ is elementary, so define $g' = \mathrm{cl}(g \cup \{\langle x, y\rangle\})$. From all the requirements in $(*)$ let us check that $\mathrm{Inv}(g') = \{0,1\}$. It is sufficient to show that $g'(b) \neq b$ for every element of $\mathrm{Dom}(g')$ of the form $b = a \cup x$, where $a \in \mathrm{Dom}(g)$ and $a \cap a_0 = 0$. Then, $g'(b) = g(a) \cup y$. Since $y \notin \bar{a}$ and $y \neq x$, $y \not\subseteq a \cup x$ so $g'(b) \neq b$.

Secondly, let us prove that if $e \in B$, $a_0 \in \mathrm{At}(\mathrm{Dom}(g))$, $e \subseteq a_0$ and both $\bar{e}$ and $\overline{a_0} - \bar{e}$ are infinite, then there is $g' \supseteq g$ such that g' satisfies $(*)$ and $\mathrm{Dom}(g') = \mathrm{cl}(\mathrm{Dom}(g) \cup \{e\})$. Let $x \in \bar{e} \cap g(\overline{a_0})$; then $g'' \overset{\mathrm{def}}{=} g \cup \{\langle e, e \cap g(a_0) - x\rangle\}$ is elementary, let $g' = \mathrm{cl}(g'')$. Again it is easy to check that g' satisfies $(*)$.

Suppose f_n has been defined and n is even. (In odd stages we will treat f_n^{-1} in the same way.) If $e \overset{\mathrm{def}}{=} e_{n/2} \in \mathrm{Dom}(f_n)$, define $f_{n+1} = f_n$. Otherwise, let $\{x_1, \cdots, x_k\} = \mathrm{At}(B) \cap \mathrm{Alg}(\mathrm{Dom}(f_n) \cup \{e\})$. Let $f'_n \supseteq f_n$ be such that: $\mathrm{Dom}(f'_n) = \mathrm{cl}(\mathrm{Dom}(f_n) \cup \{x_1, \cdots, x_n\})$ and f'_n satisfies $(*)$. This is possible by the first argument in the proof. Let $\{e^1, \cdots, e^m\} = \{e \cap a \mid a \in \mathrm{At}(\mathrm{Dom}(f'_n))$ and $0 \neq e \cap a \neq a\}$. Let $f_{n+1} \supseteq f'_n$ be such that $\mathrm{Dom}(f_{n+1}) =$

$\mathrm{cl}(\mathrm{Dom}(f_n) \cup \{e^1, \cdots, e^m\})$ and f_{n+1} satisfies (*). Such a function exists by our preceding arguments. $f = \bigcup_{n<\omega} f_n$ is the desired automorphism.

REMARK. In the same way one can construct a good automorphism f such that for every $a \in B$, if $f(a) \subsetneq a$, then $a = 0$ or $a = 1$.

The proof of (a) is almost included in the proof of (b).

Let $M_3(B) = \langle \mathrm{Aut}(B), \mathrm{At}(B), \mathrm{Al}(B), \mathcal{R}_1(\mathrm{At}(B)); \cdots \rangle$ and $K_3' = \{M_3(B) \mid B \in K_0'\}$. From 2.2(e) it follows that K_3' is explicitly interpretable in K_2'.

DEFINITION (Monk [4], McKenzie [3] independently). Let $\bar{a} \subseteq \mathrm{At}(B)$. $\bar{a}$ will be called an excellent subset of $\mathrm{At}(B)$ if $\bar{a}$ is infinite and for every $b \in B$ either $\bar{a} \cap \bar{b}$ is finite or $\bar{a} - \bar{b}$ is finite, and for every $b \in B$, $\bar{b} \cap \mathrm{At}(B) \neq \bar{a}$.

Suppose $M(B)$ is a model of the form $\langle \mathrm{Aut}(B), U_1(B), \cdots; \cdots \rangle$, and for every $f \in \mathrm{Aut}(B)$, f induces a unique automorphism f^* of $M(B)$ (this is always the case in what follows); let $\vec{a} = \langle a_1, \cdots, a_n \rangle$, $\vec{b} = \langle b_1, \cdots, b_n \rangle$ be sequences in $|M(B)|$ and $f \in \mathrm{Aut}(B)$ we denote $\vec{a} \overset{f}{\cong} \vec{b}$ if $f^*(a_i) = b_i$, $i = 1, \cdots, n$, and $\vec{a} \cong \vec{b}$ if there is g such that $\vec{a} \overset{g}{\cong} \vec{b}$.

LEMMA 2.3. (a) *If $\bar{a} \subseteq \mathrm{At}(B)$ is infinite then there is $\bar{b} \subseteq \bar{a}$ which is excellent.*

(b) (Monk) *If $\bar{a} \subseteq \mathrm{At}(B)$ is excellent and τ is a permutation of $\bar{a}$ then there is a unique automorphism $f \in \mathrm{Aut}(B)$ which extends $\pi \cup \mathrm{Id} \restriction ((\mathrm{At}(B) - \bar{a}) \cup \mathrm{Al}(B))$, and f is good.*

(c) (McKenzie) *If $\bar{a} \subseteq \mathrm{At}(B)$ and for no $b \in B^{\mathrm{AT}}$ $\bar{b} = \bar{a}$, then there is an excellent $\bar{c}$ such that $\bar{c} \cap \bar{a}$ and $\bar{c} - \bar{a}$ are both infinite.*

(d) *If $\bar{a} \subseteq \mathrm{At}(B)$ is excellent then there is an excellent $\bar{b}$ such that $\bar{b} \supseteq \bar{a}$ and $\bar{b} - \bar{a}$ is infinite.*

(e) *If $\bar{a} \subseteq \bar{b}$, $\bar{a}$ is infinite and $\bar{b}$ is excellent, then $\bar{a} \cong \bar{b}$.*

(f) *Let $\varphi_0(\bar{b})$ be the formula in the language of K_3' which says:* (i) $\bar{b} \subseteq \mathrm{At}(B)$; (ii) *there are $\bar{b}_1 \subsetneq \bar{b}_2 \subseteq \bar{b}$ and $f \in \mathrm{Aut}(B)$ such that $f(\bar{b}_1) = \bar{b}_2$. Then for every $B \in K_0'$ and $\bar{b} \in M_3(B)$: $M_3(B) \models \varphi_0[\bar{b}]$ iff $\bar{b}$ is an infinite subset of* $\mathrm{At}(B)$.

(g) *Let $\varphi_1(\bar{b})$ be the formula in the language of K_3' which says:* (i) $\bar{b} \subseteq \mathrm{At}(B)$ *and $\bar{b}$ is infinite*; (ii) *for every infinite $b_1 \subseteq \bar{b}$, $\bar{b}_1 \cong \bar{b}$. Then for every $B \in K_0'$ and $\bar{b} \in M_3(B)$: $M_3(B) \models \varphi_1[\bar{b}]$ iff $\bar{b}$ is excellent.*

(h) *Let $\varphi_2(\bar{b})$ be the formula in the language of K_3' which says:* (i) $\bar{b} \subseteq \mathrm{At}(B)$; (ii) *for every excellent $\bar{a} \subseteq \mathrm{At}(B)$ either $\bar{a} \cap \bar{b}$ is finite or $\bar{a} - \bar{b}$ is finite. Then for every $B \in K_0'$ and $\bar{b} \in M_3(B)$: $M_3(B) \models \varphi_2[\bar{b}]$ iff for some $a \in B$, $\bar{a} \cap \mathrm{At}(B) = \bar{b}$.*

PROOF. (b) appears in 1.1 [4] and is easily checked. (e) follows from (b) and (d), (a) follows from (c) and (e), but can be easily proved directly. (f) follows from (a) and (b), (g) follows from (e), and (h) follows from (c). We thus have to prove (c) and (d).

(c) Let $\{e_n \mid n \in \omega\}$ be an enumeration of B. We define by induction: a finite set of atoms $\overline{c_n}$, and an element of B, b_n, such that for no $b \in B$, $\bar{b} \cap \mathrm{At}(B) = \overline{b_n} \cap \bar{a}$. $\overline{c_0} = \varnothing$ and $b_0 = 1$. Suppose $\overline{c_n}$ and b_n have been defined. Either $b_n \cap e_n$ or $b_n - e_n$ satisfy the induction hypothesis on the b_n's; so define b_{n+1} to be either $b_n \cap e_n$ or $b_n - e_n$, in such a way that b_{n+1} will satisfy the induction hypothesis. Let $x \in \overline{b_{n+1}} \cap \bar{a} - \overline{c_n}$ and $y \in \overline{b_{n+1}} - \bar{a} - \overline{c_n}$, and let $\overline{c_{n+1}} = \overline{c_n} \cup \{x, y\}$. Let $\bar{c} = \bigcup_{n\in\omega} \bar{c}_n$, then $\bar{c}$ is as desired.

(d) Let $\{e_i \mid i \in B\}$ be an enumeration of B. We define by induction $b_n \in B$ and $\overline{c_n} \in S_\omega(\mathrm{At}(B))$, such that $\overline{b_n} \cap \bar{a}$ is infinite. $b_0 = 1$, $\overline{c_0} = \varnothing$. Suppose b_n, $\overline{c_n}$ have been defined. Let $b_{n+1} = b_n \cap e_n$, if $\overline{b_n} \cap \overline{e_n} \cap \bar{a}$ is infinite; and otherwise let $b_{n+1} = b_n - e_n$. Let $x \in \overline{b_{n+1}} - \bar{a} - \overline{c_n}$ and $\overline{c_{n+1}} = \overline{c_n} \cup \{x\}$. $\bigcup_{n\in\omega} \overline{c_n}$ is as desired.

Corollary 2.4. *Let*

$$M_4(B) = \langle \mathrm{Aut}(B), \mathrm{At}(B), \mathrm{Al}(B), \mathcal{R}_1(\mathrm{At}(B)), B^{\mathrm{AT}}; \cdots \rangle$$

and

$$K'_4 = \{M_4(B) \mid B \in K'_0\},$$

then K'_4 *is explicitly interpretable in* K'_3.

Proof. Immediate from 2.3(h).

Corollary 2.5. (a) *There is sentence* ψ *in the language of* K'_1 *such that for every* $B \in K'_0$, $\mathrm{Aut}(B) \models \psi$ *iff* $|\mathrm{At}(B)| < \aleph_0$.

(b) $\{B \mid B \in K'_0$ *and* $|\mathrm{At}(B)| < \aleph_0\}$ *is explicitly interpretable in* $\{\mathrm{Aut}(B) \mid B \in K'_0$ *and* $|\mathrm{At}(B)| < \aleph_0\}$.

(c) *If* $A, B \in K'_0 |\mathrm{At}(A)| < \aleph_0$ *and* $\mathrm{Aut}(A) \equiv \mathrm{Aut}(B)$ *then* $A \cong B$.

Proof. Immediate.

2.5(c) essentially appears in [3] and [4]. We shall now interpret $\langle \mathrm{Aut}(B), (B^{\mathrm{AT}})^{\mathrm{II}} \rangle$ in $\mathrm{Aut}(B)$. This is of course possible only for the class $K_0 = \{B \mid B \in K'_0 \text{ and } |\mathrm{At}(B)| = \aleph_0\}$. So let $K_i = \{M_i(B) \mid B \in K_0\}$, $i = 1, \cdots, 4$.

If f is a function and $x \in \mathrm{Dom}(f)$, let $\mathrm{Or}(f, x) = \min(\{i \mid 0 < i$ and $f^i(x) = x\} \cup \{\omega\})$. $\bar{a} \subseteq \mathrm{At}(B)$ is meager if for every $b \in B$ if $\bar{b} \cap \mathrm{At}(B)$ is infinite then $\bar{b} \cap \mathrm{At}(B) - \bar{a}$ is infinite.

Lemma 2.6. (a) *Let* $B \in K_0$, *then there is* $\bar{a} \subseteq \mathrm{At}(B)$ *such that both* $\bar{a}$ *and* $\mathrm{At}(B) - \bar{a}$ *are meager.*

(b) *If $B \in K_0$, $\bar{a} \subseteq \mathrm{At}(B)$ is meager, and E is an equivalence relation on $\bar{a}$ then there is a good automorphism of B such that for every $x, y \in \bar{a} : xEy$ iff $|\mathrm{Or}(f, x)| = |\mathrm{Or}(f, y)| < \aleph_0$.*

(c) *Let*

$$M_5(B) = \langle \mathrm{Aut}(B), \mathrm{At}(B), \mathrm{Al}(B), B^{\mathrm{AT}}, \mathcal{R}_1(\mathrm{At}(B)), \cdots, \mathcal{R}_n(\mathrm{At}(B)), \cdots; \cdots \rangle$$

and $K_5 = \{M_5(B) \mid B \in K_0\}$ then K_5 is explicitly interpretable in K_4.

Proof. (a) Let $\{e_i \mid i \in \omega\}$ be an enumeration of all elements e of B, such that $\bar{e} \cap \mathrm{At}(B)$ is infinite. We define by induction $\overline{a_n}, \overline{b_n} \in S_\omega(\mathrm{At}(B))$. $\overline{a_0} = \overline{b_0} = \emptyset$. If $\overline{a_n}, \overline{b_n}$ have been defined, let $x, y \in \overline{e_n} \cap \mathrm{At}(B) - (\overline{a_n} \cup \overline{b_n})$ and $x \neq y$. Let $\overline{a_{n+1}} = \overline{a_n} \cup \{x\}$ and $\overline{b_{n+1}} = \overline{b_n} \cup \{y\}$; then $\bar{a} = \bigcup_{n \in \omega} \overline{a_n}$ is as desired.

(b) As in the proof of 2.2(b) we can w.l.o.g. assume that B is atomic. Let $\{e_i \mid i \in \omega\}$ be an enumeration of B, and $\{\overline{d_i} \mid i \in \omega\}$ be a 1–1 enumeration of the equivalence classes of E. We define by induction functions f_n. Our induction hypothesis is: $(*)$ f_n is elementary, $\mathrm{Dom}(f_n) = \mathrm{Alg}(\mathrm{Dom}(f_n))$ is finite; if $c \in \mathrm{At}(\mathrm{Dom}(f_n))$ and $\bar{c}$ is infinite then $f_n(c) = c$; if $x \in \overline{d_i} \cap \mathrm{Dom}(f_n)$ then $f_n(x)$, $f_n^2(x), \cdots, f_n^{i+1}(x)$ are all defined, and $i + 1$ is the first positive integer l for which $f^l(x) = x$.

Note that $(*)$ implies f_n is good. We first prove the following facts. (1) If g satisfies $(*)$ and $x \in \overline{d_i}$, then there is $\bar{g} \supseteq g$ which satisfies $(*)$, $x \in \mathrm{Dom}(\bar{g})$ and $\mathrm{Dom}(\bar{g}) = \mathrm{cl}(\mathrm{Dom}(g) \cup \{x, \bar{g}(x), \cdots, \bar{g}^i(x)\})$. (2) If g satisfies $(*)$, $b \in B$, and either $b \in \mathrm{At}(B) - \bar{a}$ or there is $c \in \mathrm{At}(\mathrm{Dom}(g))$ such that $b \subseteq c$, and $\bar{b}$ and $\bar{c} - \bar{b}$ are infinite, then $g \cup \{\langle b, b\rangle\}$ is elementary, and $\mathrm{cl}(g \cup \{\langle b, b\rangle\})$ satisfies $(*)$.

Proof of (1). Let $x \subseteq c \in \mathrm{At}(\mathrm{Dom}(g))$. If $\bar{c}$ is finite, then since $\mathrm{Dom}(g) = \mathrm{Alg}(\mathrm{Dom}(g))$, $x \in \mathrm{Dom}(g)$; so, we define $\bar{g} = g$. If $\bar{c}$ is infinite, then $\bar{c} - \bar{a}$ is infinite. Let $x_1, \cdots, x_n \in \bar{c} - \bar{a}$ be distinct. $g(c) = c$, so $g' \overset{\mathrm{def}}{=} g \cup \{\langle x, x_1\rangle, \langle x_1, x_2\rangle, \cdots, \langle x_{i-1}, x_i\rangle, \langle x_i, x\rangle\}$ is elementary. Let $\bar{g} = \mathrm{cl}(g')$, it is easy to check that $\bar{g}$ is as required.

(2) is trivial. The rest of the proof resembles the last part in the proof of 2.2(b).

Proof of (c). It is a well-known fact that if $\mathrm{Eq}(A)$ is the set of equivalence relations on A then the class $\{M^{\mathrm{II}} \mid M$ is an L-model$\}$ is explicitly interpretable in the class $\{\langle M, \mathrm{Eq}(|M|); E_2 \rangle \mid M$ is an L-model$\}$ where $\langle a, b, e\rangle \in E_2$ iff $e \in \mathrm{Eq}(|M|)$ and $\langle a, b\rangle \in e$. Thus in order to prove (c) it suffices to show how to interpret equivalence relations on $\mathrm{At}(B)$ in $M_4(B)$. We represent $\mathrm{Eq}(\mathrm{At}(B))$ by the set $\{\langle \bar{a}, f, g\rangle \mid \bar{a} \subseteq \mathrm{At}(B),\ f, g \in \mathrm{Aut}(B)\}$;

$\langle \bar{a}, f, g\rangle = \tau$ represents the equivalence relation E_τ which is defined as follows: $\langle x, y\rangle \in E_\tau$ iff $y, x \in \bar{a}$ and $|\mathrm{Or}(f, x)| = |\mathrm{Or}(f, y)|$, or $x, y \in \mathrm{At}(B) - \bar{a}$ and

$|\mathrm{Or}(g,x)| = |\mathrm{Or}(g,y)|$, or $x \in \bar{a}$ and $y \in \mathrm{At}(B) - \bar{a}$, $|\mathrm{Or}(f,x)| = |\mathrm{Or}(g,y)|$, or $x \in \mathrm{At}(B) - \bar{a}$ and $y \in \bar{a}$ and $|\mathrm{Or}(g,x)| = |\mathrm{Or}(f,y)|$. Certainly there is a formula $\varphi_3(x, y, \bar{a}, f, g)$ in the language of K_4 such that for every $B \in K_0$ $x, y, \bar{a}, f, g \in |M_4(B)|$:

$$M_4(B) \models \varphi_3[x, y, \bar{a}, f, g] \quad \text{iff} \quad \langle x, y \rangle \in E_{\langle \bar{a}, f, g \rangle}.$$

By 2.6(a) and (b) for every $E \in \mathrm{Eq}(\mathrm{At}(B))$ there are $\bar{a}$, f and g such that $E = E_{\langle \bar{a}, f, g \rangle}$. Hence $\mathrm{Eq}(\mathrm{At}(B))$ can be interpreted in $M_4(B)$, and the claim of the lemma follows.

LEMMA 2.7. *Let L be any first order language,*

$$\bar{K}_1 = \{\langle M^{\mathrm{II}}, U_1, \cdots, U_n, \cdots; E_1, \cdots, E_n, \cdots \rangle \mid M \text{ is an infinite } L\text{-model}, \; U_i \subseteq \mathcal{R}_i(|M|) \text{ and } |U_i| \leqq \|M\| \; i = 1, 2, \cdots\}$$

and let

$$\bar{K}_2 = \{\langle M, U_1, \cdots, U_n, \cdots; E_1, \cdots, E_n, \cdots \rangle^{\mathrm{II}} \mid M \text{ and } U_i \text{ are as above}\}.$$

Then $\bar{K}_2$ is explicitly interpretable in $\bar{K}_1$.

PROOF. Let R be a binary relation on the universe of $\langle M, U_1, \cdots, U_n, \cdots; E_1, \cdots, E_n, \cdots \rangle$, R can be encoded by a set

$$S_R \subseteq |M| \times \bigcup_{n,k \in \omega} (|M|^n \times |M|^k) = \mathcal{M}$$

in the following way: $\langle r_1, r_2 \rangle \in R$ iff there exists an $x \in |M|$ such that $\{\langle \bar{a}, \bar{b} \rangle \mid \langle x, \bar{a}, \bar{b} \rangle \in S_R\} = r_1 \times r_2$. Since $\langle \mathcal{M}, \mathcal{R}_1(\mathcal{M}); \cdots \rangle$ can be interpreted in M^{II} the lemma follows.

COROLLARY 2.8. *Let $M_6(B) = \langle \mathrm{Aut}(B); (B^{\mathrm{AT}\,\mathrm{II}}, \mathrm{Al}(B); \mathrm{Op}, \cdots \rangle$ and $K_6 = \{M_6(B) \mid B \in K_0\}$, then K_6 is explicitly interpretable in K_5.*

PROOF. Immediate from 2.7, for take $U_1 = \{\bar{b} \mid b \in B^{\mathrm{AT}}\}$ and $U_i = \varnothing$ for $i > 1$.

We shall now show how $\langle B^{\mathrm{AL}}, \mathrm{Al}(B); \cdots \rangle$ can be interpreted in $\mathrm{Aut}(B)$.

Let $\bar{a} \subseteq \mathrm{Al}(B)$ and $b \in B$ then we define $\bar{a} \restriction b = \{c \mid c \in \bar{a} \text{ and } c \cap b \neq 0\}$.

DEFINITIONS. Let $\bar{a} \subseteq \mathrm{Al}(B)$; $\bar{a}$ will be called an excellent subset of $\mathrm{Al}(B)$ if $\bar{a}$ is an infinite set of pairwise disjoint elements, for every $b \in B$ either $|\bar{a} \restriction b| < \aleph_0$ or $|\bar{a} \restriction -b| < \aleph_0$, and if $b \in B$ and $\bar{a} \restriction b$ is infinite, then there is $c \in \mathrm{Al}(B)$ such that $c \subseteq b$ and $c \cap d = 0$ for every $d \in \bar{a}$.

Let $\bar{a}, \bar{b} \subseteq \mathrm{Al}(B)$ be sets of pairwise disjoint elements, $\bar{b}$ is called a refinement of $\bar{a}$ if for every $c \in \bar{b}$ there is $d \in \bar{a}$ such that $c \subseteq d$, and if $c_1, c_2 \in \bar{b}$, $d \in \bar{a}$ and $c_1, c_2 \subseteq d$, then $c_1 = c_2$.

LEMMA 2.9. (a) *If* $\bar{a} \subseteq \mathrm{Al}(B)$ *is excellent then there is* $f \in \mathrm{Aut}(B)$ *such that* $\bar{a} = \mathrm{At}(\mathrm{Inv}(f)) \cap \mathrm{Al}(B)$.

(b) *If* $\bar{b} \subseteq \mathrm{Al}(B)$ *is an infinite set of pairwise disjoint elements, then* $\bar{b}$ *has a refinement which is excellent.*

(c) *Let* $\bar{a} \subseteq \mathrm{Al}(B)$ *be excellent and* f *be an elementary function with the following properties*:

(i) $\mathrm{Dom}(f) = \mathrm{Rng}(f) = \{c \mid \text{for some } d \in \bar{a}\ c \subseteq d\}$;

(ii) *for every* $d \in \bar{a}$ *there is* $e \in \bar{a}$ *such that* $f \restriction \{c \mid c \subseteq d\}$ *is an elementary function onto* $\{c \mid c \subseteq e\}$.

Then f *can be uniquely extended to an automorphism* f *of* B *such that for every* c: *if* $c \cap d = 0$ *for every* $d \in \bar{a}$, *then* $f(c) = c$.

(d) *If* $\bar{a} \subseteq \mathrm{Al}(B)$ *is excellent then there is an excellent* $\bar{b}$ *such that for every* $c \in \bar{a}$ *there is* $d \in \bar{b}$ *such that* $c \subseteq d$ *and* $d - c \neq 0$ *and* $|\{d \mid d \in \bar{b} \text{ and for no } c \in \bar{a}\ c \subseteq d\}| = \aleph_0$.

(e) *Let* $f \in \mathrm{Aut}(B)$, *then for no* $b \in B$ $\mathrm{fix}(f) \cap \mathrm{Al}(B) = \bar{b} \cap \mathrm{Al}(B)$ *iff there is an excellent* $\bar{a} \subseteq \mathrm{Al}(B)$ *such that* $|\bar{a} \cap \mathrm{fix}(f)| = \aleph_0$ *and* $|\bar{a} \cap \mathrm{var}(f)| = \aleph_0$, *where* $\mathrm{var}(f) = \{c \mid x \in \mathrm{At}(B) \cup \mathrm{Al}(B) \text{ and for every } d \in \mathrm{fix}(f)\ d \cap c = 0\}$.

(f) *If* $\bar{a} \subseteq \mathrm{Al}(B)$ *is excellent and* $\bar{b}$ *is an infinite refinement of* $\bar{a}$, *then* $\bar{a} \cong \bar{b}$; *that is, there is* $f \in \mathrm{Aut}(B)$ *such that* $f(\bar{a}) = \bar{b}$.

(g) *Let* $I \subseteq \mathrm{Al}(B) \cup \{0\}$ *be a complete ideal relative to* $\mathrm{Al}(B)$ (*that is,* I *is an ideal in* B; *and for every* $b \in \mathrm{Al}(B)$, *if for every* $c \in \bar{b}$ $\bar{c} \cap I \neq \emptyset$, *then* $b \in I$); *then there is* $f \in \mathrm{Aut}(B)$ *such that* $\mathrm{fix}(f) = (I - \{0\}) \cup \mathrm{At}(B)$.

PROOF. The proof of (c) is immediate.

Proof of (a). For every $c \in \bar{a}$ let f_c be an elementary function from $\bar{c}$ onto $\bar{c}$ such that $\mathrm{Inv}(f_c) = \{0, c\}$. Let $f = \bigcup\{f_c \mid c \in \bar{a}\}$ and let $\tilde{f}$ be an extension of f as in (c) then $\mathrm{At}(\mathrm{Inv}(\tilde{f})) \cap \mathrm{Al}(B) = \bar{a}$.

Proof of (b). Let $\{b_i \mid i \in \omega\}$ be an enumeration of B. We define by induction $c_i \in B$ and $d_i \in \mathrm{Al}(B)$ such that

(i) for every $i > j$, $c_i \subseteq c_j$;

(ii) for every i, $\bar{b} \restriction (c_i - \bigcup_{j \leq i} d_j)$ is infinite.

Let $c_0 = 1$ and d_0 be some element of $\bar{b}$. Suppose c_j, d_j were defined for every $j \leq i$; $\bar{b} \restriction (c_i - \bigcup_{j \leq i} d_j)$ is infinite so either $\bar{b} \restriction (c_i \cap b_i - \bigcup_{j \leq i} d_j)$ or $\bar{b} \restriction (c_i - b_i - \bigcup_{j \leq i} d_j)$ is infinite. W.l.o.g. suppose the first one is infinite and let $c_{i+1} = c_i \cap b_i - \bigcup_{j \leq i} d_j$. Let $d \in \bar{b}$ such that for no $j \leq i$, $d_j \subseteq d$ and $d \cap c_{i+1} \neq 0$ and let $\bar{d}_{i+1} \subsetneqq d \cap c_{i+1}$. It is easy to check that $\{d_i \mid i \in \omega\}$ is an excellent refinement of $\bar{b}$.

The proof of (d) and (e) is by a construction similar to the construction in (b), and (f) is a corollary of (c) and (d).

Proof of (g). It is sufficient to construct $f \in \mathrm{Aut}(B^{\mathrm{AL}})$ such that $\mathrm{fix}(f) = I - \{0\}$, and for every $b \in B^{\mathrm{AL}}$ $(b - f(b)) \cup (f(b) - b) \in \mathrm{Al}(B) \cup \{0\}$; since, if f is such an automorphism, then $f \cup \mathrm{Id} \restriction \mathrm{At}(B)$ induces an automorphism of B which is as required.

Let $\{e_i \mid i \in \omega\}$ be an enumeration of B^{AL}. We define by induction finite elementary functions f_n, which satisfy the following induction hypotheses: (1) $\mathrm{Dom}(f_n) \subseteq B^{\mathrm{AL}}$; (2) $f_n = \mathrm{cl}(f_n)$; (3) if $a \in \mathrm{At}(\mathrm{Dom}(f_n))$ and $\bar{a} \cap I \neq \emptyset$ or $a \notin \mathrm{Al}(B)$ then $f_n(a) = a$; (4) if $a \in \mathrm{At}(\mathrm{Dom}(f_n))$, then either $f_n(a) = a$ or $a \cap f_n(a) = 0$.

Let $f_0 = \{\langle 0, 0\rangle, \langle 1^{B^{\mathrm{AL}}}, 1^{B^{\mathrm{AL}}}\rangle\}$. Suppose f_n has been defined. We define f_{n+1} so that its domain and range will contain e_n. W.l.o.g. there is $a \in \mathrm{At}(\mathrm{Dom}(f_n))$ such that $e_n \subseteq a$ (because we can add e_n to the domain of f_{n+1} piece by piece). If $\bar{e}_n \cap I \neq \emptyset$ or $e_n \notin \mathrm{Al}(B)$, then $f_n(a) = a$, so $f_n \cup \{\langle e_n, e_n\rangle\}$ is elementary, and its closure satisfies the induction hypotheses.

Suppose $e_n \in \mathrm{Al}(B)$ and $\bar{e}_n \cap I = \emptyset$. W.l.o.g. $e_n \neq a$. If $f_n(a) \cap a = 0$ let $c \in f_n(\bar{a})$, then $\mathrm{cl}(f_n \cup \{\langle e_n, c\rangle\})$ satisfies the induction hypotheses. If $f_n(a) = a$ let $c \in \bar{e}_n$, then $f_n \cup \{\langle e_n - c, c\rangle, \langle c, e_n - c\rangle\}$ is elementary, and its closure satisfies the induction hypotheses. After adding e_n to $\mathrm{Dom}(f_{n+1})$ we add it to $\mathrm{Rng}(f_{n+1})$ in a similar way. $\bigcup_{n\in\omega} f_n$ is as desired. Q.E.D.

Let $M_7(B) = \langle M_6(B), \mathrm{FIX}^{\mathrm{AL}}(B), \mathrm{AV}^{\mathrm{AL}}(B); \in\rangle$, where $\mathrm{FIX}^{\mathrm{AL}}(B) = \{\mathrm{fix}(f) \cap \mathrm{Al}(B) \mid f \in \mathrm{Aut}(B)\}$, $\mathrm{AV}^{\mathrm{AL}}(B) = \{\mathrm{At}(\mathrm{Inv}(f)) \cap \mathrm{Al}(B) \mid f \in \mathrm{Aut}(B)\}$ and $\in$ is the belonging relation between elements of $\mathrm{Al}(B)$ and elements of $\mathrm{FIX}^{\mathrm{AL}}(B) \cup \mathrm{AV}^{\mathrm{AL}}(B)$. Let $K_7 = \{M_7(B) \mid B \in K_0\}$, then certainly K_7 is explicitly interpretable in K_6.

LEMMA 2.10. (a) *Let $\varphi_4(\bar{a})$ be the formula in the language of K_7 which says*:

(i) $\bar{a} \in \mathrm{AV}^{\mathrm{AL}}(B)$;

(ii) *there is a refinement $\bar{b}$ of $\bar{a}$ and $\bar{b}' \subsetneqq \bar{b}$ such that $\bar{b} \cong \bar{b}'$. Then for every $B \in K_0$ and $\bar{a} \in |M_7(B)|$: $M_7(B) \models \varphi_4[\bar{a}]$ iff $\bar{a}$ is an infinite element of* $\mathrm{AV}^{\mathrm{AL}}(B)$.

(b) *Let $\varphi_5(\bar{a})$ be the formula in the language of K_7 which says that $\bar{a} \subseteq \mathrm{AV}^{\mathrm{AL}}(B)$, $\bar{a}$ is infinite, and if $\bar{b}$ is an infinite refinement of $\bar{a}$ then $\bar{b} \cong \bar{a}$. Then for every $B \in K_0$ and $\bar{a} \in |M_7(B)|$: $M_7(B) \models \varphi_5[\bar{a}]$ iff $\bar{a}$ is an excellent subset of* $\mathrm{Al}(B)$.

(c) *Let $\varphi_6(\bar{a})$ be the formula in the language of K_7 which says: $\bar{a} \in \mathrm{FIX}^{\mathrm{AL}}(B)$ and for no excellent subset $\bar{b} \subseteq \mathrm{Al}(B)$ $\bar{b} \cap \bar{a}$ is infinite and $\bar{b} \cap \{c \mid c \in \mathrm{Al}(B)$ and* *for every* $d \in \bar{a}$ $d \cap c = 0\}$ *is infinite. Then for every* $B \in K_0$ *and* $\bar{a} \in |M_7(B)|$: $M_7(B) \models \varphi_6[\bar{a}]$ *iff* $\bar{a} \in \mathrm{FIX}^{\mathrm{AL}}(B)$ *and for some* $b \in B$ $\bar{a} = \bar{b} \cap \mathrm{Al}(B)$.

(d) *Let* $M_8(B) = \langle \mathrm{Aut}(B), B^{\mathrm{AT}}, B^{\mathrm{AL}}, \mathrm{Al}(B), \cdots, \mathcal{R}_n(B^{\mathrm{AT}}), \cdots; \cdots \rangle$ *and* $K_8 = \{M_8(B) \mid B \in K_0\}$ *then* K_8 *is explicitly interpretable in* K_7.

PROOF. Immediate from 2.9. Notice that if $b \in B$ then $\bar{b} \cap \mathrm{Al}(B)$ is a complete ideal relative to $\mathrm{Al}(B)$.

REMARK. Notice that in general it is impossible to interpret B in $M_8(B)$ since the relation $N \subseteq B^{\mathrm{AT}} \times B^{\mathrm{AL}}$, $N = \{\langle a, b \rangle \mid a \cup b \in B\}$ is not one of the relations included in $M_8(B)$.

LEMMA 2.11. *Let* $M_9(B) = \langle M_8(B), \cdots, \mathcal{R}_n^{<\omega}(B^{\mathrm{AL}}), \cdots; \cdots, E_n, \cdots \rangle$ *where* $\mathcal{R}_n^{<\omega}(C)$ *is the set of all finite n-place relations whose domain is a subset of C, and let* $K_9 = \{M_9(B) \mid B \in K_0\}$ *then* K_9 *is explicitly interpretable in* K_8.

PROOF. If $C \subseteq B^{\mathrm{AL}}$ is a finite set of non-zero pairwise disjoint elements then there are $f, g \in \mathrm{Aut}(B)$ such that $C = \mathrm{At}(\mathrm{Inv}(f)) \cap \mathrm{At}(\mathrm{Inv}(g))$. Let $C \subseteq B^{\mathrm{AL}}$ be a set of non-zero pairwise disjoint elements, then C is finite iff there is an excellent set $\bar{a} \subseteq \mathrm{Al}(B)$ such that for every $c \in C$ there is $d \in \bar{a}$ such that $d \subseteq c$ and there is $c_0 \in C$ such that $\{d \mid d \in \bar{a}$ and for some $c \in C - \{c_0\} d \subseteq c\} \not\approx \bar{a}$. So finite sets of pairwise disjoint elements of B^{AL} can be interpreted. Let C_1, C_2 be finite sets of pairwise disjoint elements and $D(C_1, C_2) = \{d(C_1, c) \mid c \in C_2\}$ where $d(C_1, c) = \cup\{c' \mid c' \in C_1$ and $c' \cap c \neq 0\}$. $D(C_1, C_2)$ can be defined in $M_8(B)$ from C_1 and C_2 and every finite set of non-zero elements of B^{AL} is $D(C_1, C_2)$ for some C_1 and C_2, so arbitrary finite subsets of B^{AL} can be interpreted in $M_8(B)$. Finite symmetric relations can be defined similarly, and the lemma follows.

If $f_1, \cdots, f_n$ are automorphisms of B and $C \subseteq B^{\mathrm{AL}}$, let $\mathrm{cl}(f_1, \cdots, f_n, C)$ be the smallest subset of B^{AL} which includes C and is closed under $f_1, \cdots, f_n, f_1^{-1}, \cdots, f_n^{-1}$.

LEMMA 2.12. (a) *In* $M_9(B)$ $\mathrm{cl}(f_1, \cdots, f_n, C)$ *is definable from* $f_1, \cdots, f_n, C$.

(b) *If* $C \subseteq B^{\mathrm{AL}}$ *then the closure of* C *in* B^{AL} *is definable from* C *in* $M_9(B)$.

PROOF. The proof is simple.

DEFINITION. Let $C \subseteq B^{\mathrm{AL}}$, C will be called a good generating chain (GGC), if for every $c, d \in C$ either $c \subseteq d$ or $d \subseteq c$; $\mathrm{cl}(C) = B^{\mathrm{AL}}$; and for every $c \in C$, $\langle \{d \mid d \in C$ and $(d - c) \cup (c - d) \in \mathrm{Al}(B) \cup \{0\}\}, \subseteq \rangle$ has the order type of the rationals.

LEMMA 2.13. (a) *If* $B^{AL} \neq \{0\}$ *then there is a GGC* $C \subsetneq B^{AL}$.

(b) *If* $C \subsetneq B^{AL}$ *is a GGC and f is an automorphism of* $\langle C, \subsetneq \rangle$ *such that for every* $c \in C$ $(c - f(c)) \cup (f(c) - c) \in \mathrm{Al}(B) \cup \{0\}$, *then* $f \cup (\mathrm{Id} \restriction \mathrm{At}(B))$ *can be uniquely extended to an automorphism of B.*

PROOF. (a) is proved by a simple inductive construction, and (b) is immediate.

LEMMA 2.14. (a) *If* $C \subsetneq B^{AL}$ *is a GGC then there are* $f_1, f_2, f_3, f_4 \in \mathrm{Aut}(B)$ *such that* $C = \mathrm{cl}(f_1, f_2, \{c \mid c \in B^{AL}, f_3(c) = c$ *and* $f_4(c) \supsetneq c\})$.

(b) *Let* $M_{10}(B) = \langle M_9(B), \{C \mid C \subsetneq B^{AL}$ *and C is a GGC*$\}; \in \rangle$ *and* $K_{10} = \{M_{10}(B) \mid B \in K_0\}$ *then* K_{10} *is explicitly interpretable in* K_9.

PROOF. (a) Let $D \subsetneq C$ be a maximal set with the property that if $d_1, d_2 \in D$ then $d_1 - d_2 \notin \mathrm{Al}(B)$. Let g_3 be an automorphism of $\langle C, \subsetneq \rangle$ such that $d \in C - D$ iff $g_3(d) - d \in \mathrm{Al}(B)$. Let g_4 be an automorphism of $\langle C, \subsetneq \rangle$ such that for every $c \in C$ $g_4(c) - c \in \mathrm{Al}(B)$. Let g_1, g_2 be automorphisms of $\langle C, \subsetneq \rangle$ such that $\mathrm{cl}(g_1, g_2, D) = C$ and for every $c \in C$ or $(g_i(c) - c) \cup (c - g_i(c)) \in \mathrm{Al}(B) \cup \{0\}$, $i = 1, 2$. Let f_i be the extension $g_i \cup (\mathrm{Id} \restriction \mathrm{At}(B))$ to an automorphism of B, $i = 1, \cdots, 4$; then $f_1, \cdots, f_4$ are as desired.

(b) is trivial from (a).

2.15(d) is the strongest interpretability result that we obtain for the class K_0.

LEMMA 2.15. (a) *Let* $C \subsetneq B^{AL}$ *be a GGC,* $C_1 \subsetneq C$ *be dense in* $\langle C, \subsetneq \rangle$ *and* $D \subsetneq C_1$ *be a maximal set with the property that for every* $d_1, d_2 \in D$, $d_1 - d_2 \notin \mathrm{Al}(B)$. *Then there are* f_1, f_2 *automorphisms of* $\langle C, \subsetneq \rangle$ *such that for every* $c \in C$

$$(f_i(c) - c) \cup (c - f_i(c)) \in \mathrm{Al}(B) \cup \{0\}, \quad i = 1, 2 \quad \text{and} \quad C_1 = \mathrm{cl}(f_1, f_2, D).$$

(b) *Let* $C \subsetneq B^{AL}$ *be a GGC,* $C_1 \subsetneq C$ *be dense and codense in* $\langle C, \subsetneq \rangle$ *and E be an equivalence relation on* C_1. *Then there is an automorphism f of* $\langle C, \subsetneq \rangle$ *such that for every* $c \in C$ $(f(c) - c) \cup (c - f(c)) \in \mathrm{Al}(B) \cup \{0\}$ *and for every* $c_1, c_2 \in C_1$, $\langle c_1, c_2 \rangle \in E$ *iff* $|D(C_1, c_1, f)| = |D(C_1, c_2, f)| < \aleph_0$, *where*

$$D(C_1, c_i, f) = \{c \mid (\exists n > 0)(f^n(c_i) = c \wedge (\forall k \leqq n)(k > 0 \rightarrow f^k(c_i) \in C - C_1))\}.$$

(c) *Let* $M_{11}(B) = \langle M_{10}(B), \cup \{\mathrm{Eq}(C) \mid C$ *is a GGC in* $B\}; E_2 \rangle$, *where* $\mathrm{Eq}(C)$ *is the set of equivalence relations on C, and* $\langle a, b, e \rangle \in E_2$ *iff e is an equivalence relation on some GGC and* $\langle a, b \rangle \in e$; *let* $K_{11} = \{M_{11}(B) \mid B \in K_0\}$. *Then* K_{11} *is explicitly interpretable in* K_{10}.

(d) *Let* $M_{12}(B) = \langle \mathrm{Aut}(B), (B^{AT})^{II}, \langle B^{AL}, \mathrm{Al}(B), \cdots \rangle^{II}; \mathrm{Op} \rangle$ *and* $K_{12} = \{M_{12}(B) \mid B \in K_0\}$ *then* K_{12} *is explicitly interpretable in* K_{11}.

PROOF. (a) and (b) are proved by a simple inductive construction.

(c) For every $\langle C, C_0, f_0, f_1 \rangle = \tau$ such that: C is a GGC, C_0 is dense and codense in C, $f_0, f_1 \in \mathrm{Aut}(B)$, and $f_i \restriction C$ is an automorphism of $\langle C, \subseteq \rangle$ such that for every $c \in C$ $(f_i(c) - c) \cup (c - f_i(c)) \in \mathrm{Al}(B) \cup \{0\}$, $i = 0, 1$; let $C_1 = C - C_0$ and $E_\tau = \{\langle c_1, c_2 \rangle \mid |D(C_i, c_1, f_i)| = |D(C_j, c_2, f_j)|$, where $c_1 \in C_i$ and $c_2 \in C_j\}$, then certainly dense and codense subsets of a GGC can be interpreted in $M_{10}(B)$ and E is definable from τ in $M_{10}(B)$.

It follows from (b) that when τ ranges over all quadruples of the above form, E_τ ranges over all equivalence relations on GGC's. So (c) is proved.

(d) If $C \subseteq B^{AL}$ is a GGC then $\mathrm{cl}(C) = B^{AL}$ so $(B^{AL})^{II}$ can be interpreted in $\langle C, \subseteq \rangle^{II}$. (d) follows easily from this fact.

Let $\mathrm{As}(B)$ be the set of atomic elements of B, and $I(B) = \{a \cup b \mid a \in \mathrm{As}(B), b \in \mathrm{Al}(B)\}$.

LEMMA 2.16. (a) *If* $B/I(B) \neq \{0\}$ *and* $B_1/I(B_1) \equiv B/I(B)$ *then* $B \equiv B_1$.
(b) *For every BA,* $C/I(C) \cong C^{AL}/\mathrm{Al}(C) \cup \{0\} \cong C^{AT}/\mathrm{As}(C)$.

PROOF. (a) Is well known, see e.g. [1] pp. 293–302.
The proof of (b) is easy.

COROLLARY 2.17. *If* $A, B \in K_0'$ *and* $\mathrm{Aut}(A) \equiv \mathrm{Aut}(B)$ *then* $A \equiv B$. *Moreover for every complete theory of BA's* T *there is a sentence* φ_T *in the language of groups such that for every* $A \in K_0'$ $A \models T$ *iff* $\mathrm{Aut}(A) \models \varphi_T$.

PROOF. The first part of 2.17 follows from the fact $\langle B^{AT}, B^{AL}, \mathrm{Al}(B); \cdots \rangle$ can be interpreted in $\mathrm{Aut}(B)$. The second part of 2.17 follows easily from the fact that for every complete theory of BA's T there is a second order sentence ψ_T such that for every BA $C : C \models T$ iff $C \models \psi_T$.

COROLLARY 2.18. $\{B^{II} \mid B \in K_0'$ *and* B *has a maximal atomic element*$\}$ *is explicitly interpretable in* $\{\mathrm{Aut}(B) \mid B \in K_0'$ *and* B *has a maximal atomic element*$\}$.

The following theorem appears in [2].

THEOREM 2.19. *Assume* $V = L$. *If* M, N *are countable models in the same finite language and* $M^{II} \equiv N^{II}$, *then* $M \cong N$.

COROLLARY 2.20. $(V = L)$ *If* $A, B \in K_0'$, A *has a maximal atomic element* *and* $\operatorname{Aut}(A) \equiv \operatorname{Aut}(B)$ *then* $A \cong B$.

PROOF. Immediate from 2.17, 2.18 and 2.19.

§3. This section is devoted to the proof of the following theorem.

THEOREM 3.1. ($V = L$) *If* A *and* B *are countable BA's, and* $\operatorname{Aut}(A) \equiv \operatorname{Aut}(B)$, *then* $\operatorname{Aut}(A) \cong \operatorname{Aut}(B)$.

Let us explain the proof. The first tendency would be, of course, to try to prove that $\{B^{\mathrm{II}} \mid B \text{ is countable}\}$ is explicitly interpretable in $\{\operatorname{Aut}(B) \mid B \text{ is countable}\}$. Then applying 2.19, we would be able to conclude: $\operatorname{Aut}(A) \equiv \operatorname{Aut}(B) \Rightarrow A^{\mathrm{II}} \equiv B^{\mathrm{II}} \Rightarrow A \cong B \Rightarrow \operatorname{Aut}(A) \cong \operatorname{Aut}(B)$. However, by McKenzie's and Shelah's example (see [3]) $\operatorname{Aut}(A) \cong \operatorname{Aut}(B) \not\Rightarrow A \cong B$, so this way is wrong. However this can be amended in the following way. We will find a countable structure $M(B)$, depending on B, such that: $M(A) \cong M(B) \Rightarrow \operatorname{Aut}(A) \cong \operatorname{Aut}(B)$. (But $M(A) \cong M(B) \not\Rightarrow A \cong B$.)

Now we shall prove that $\operatorname{Aut}(A) \equiv \operatorname{Aut}(B) \Rightarrow M(A)^{\mathrm{II}} \equiv M(B)^{\mathrm{II}} \Rightarrow M(A) \cong M(B) \Rightarrow \operatorname{Aut}(A) \cong \operatorname{Aut}(B)$. (In fact the situation will be somewhat more complicated.)

We recall that $B^{\mathrm{TL}} = B^{\mathrm{AT}} \times B^{\mathrm{AL}}$, and that $\operatorname{Aut}(B)$ is regarded as a subgroup of $\operatorname{Aut}(B^{\mathrm{TL}})$. Let $\operatorname{Aut}_0(B) = \{f \circ g \mid f, g \in \operatorname{Aut}(B),\ f \restriction \operatorname{At}(B) = \mathrm{Id} \text{ and } g \restriction \operatorname{Al}(B) = \mathrm{Id}$. $\operatorname{Aut}_0(B)$ is definable in $\operatorname{Aut}(B)$, so it is normal.

DEFINITION. (a) $a \in B^{\mathrm{AT}} \cup B^{\mathrm{AL}}$ is recognizably small (RS), if there is $f \in \operatorname{Aut}_0(B)$, such that $f(a) \cap a = 0$. Denote by $\mathrm{RS}(B)$ (and when no ambiguity may arise, by RS) the set of recognizably small elements of B^{TL}.

(b) $a \in B^{\mathrm{AT}} \cup B^{\mathrm{AL}}$ is recognizably big (RB), if $a \notin \mathrm{RS}$ and there is $f \in \operatorname{Aut}(B)$, such that $f(a) \cap a = 0$. $\mathrm{RB}(B)$ and $a \in \mathrm{RB}$ are understood in the obvious way.

(c) $a \in \mathrm{RB}$ is called an atom of RB ($a \in \mathrm{ARB}$), if for every $c \subseteq a$ either $c \notin \mathrm{RB}$ or $a - c \notin \mathrm{RB}$.

(d) We define the neighboring relation on $\mathrm{RB}(B)$; it will be denoted by $\mathrm{Nb}(B)$, or, in short, by Nb. $\langle a, b \rangle \in \mathrm{Nb}$ if $a, b \in \mathrm{RB}$, and $a - b, b - a \in \mathrm{RS}$.

(e) We define the relation: "a and b are twins"; it will be denoted by $\mathrm{Tn}(B)$, or, in short, by Tn. $\langle a, b \rangle \in \mathrm{Tn}$ if: (1) $a \in \mathrm{RB} \cap B^{\mathrm{AT}}$ and $b \in \mathrm{RB} \cap B^{\mathrm{AL}}$; and (2) for every $f \in \operatorname{Aut}(B)$, $f(a) \cap a \in \mathrm{RS}$ iff $f(b) \cap b \in \mathrm{RS}$.

In the following lemma, we summarize some trivial observations.

LEMMA 3.2. (a) $a \in \mathrm{RS}$ *iff* $a \in \operatorname{As}(B) \cup \operatorname{Al}(B)$, *and for some* $f \in \operatorname{Aut}(B)$, $f(a) \cap a = 0$.

(b) $a \in \mathrm{RB}$ *iff* $a \in (B^{\mathrm{AT}} \cup B^{\mathrm{AL}}) - (\mathrm{As}(B) \cup \mathrm{Al}(B))$, *and for some* $f \in \mathrm{Aut}(B)$, $f(a) \cap a = 0$. *If* $a \in \mathrm{RB}$ *and* $c \subseteq a$, *then either* $c \in \mathrm{RB}$, *or* $c \in \mathrm{RS}$.

(c) $\langle a, b \rangle \in \mathrm{Nb}$ *iff* $a, b \in \mathrm{RB}$, *and* $(a - b) \cup (b - a) \in I(B)$; *so obviously*, Nb *is an equivalence relation.*

(d) *If* $a \in \mathrm{RB} \cap B^{\mathrm{AT}}$, $b \in \mathrm{RB} \cap B^{\mathrm{AL}}$ *and* $a \cup b \in B$, *then* $\langle a, b \rangle \in \mathrm{Tn}$. *If* $\langle a, a' \rangle$, $\langle b, b' \rangle \in \mathrm{Nb}$, *then* $\langle a, b \rangle \in \mathrm{Tn}$ *iff* $\langle a', b' \rangle \in \mathrm{Tn}$.

(e) *If* $a \in \mathrm{RB} \cap B^{\mathrm{AT}}$, *then there is* $b \in \mathrm{RB} \cap B^{\mathrm{AL}}$, *such that* $a \cup b \in B$. *The analogous fact for* B^{AL} *also holds.*

(f) *The following are equivalent*: (1) RB/Nb *is infinite*; (2) RB *contains an infinite subset of pairwise disjoint elements*; (3) $\mathrm{Aut}(B)/\mathrm{Aut}_0(B)$ *is infinite.*

(g) *If* RB/Nb *is infinite, then there are sequences* $\{a_i \mid i \in \omega\}$ *and* $\{b_i \mid i \in \omega\}$, *such that for every* $i, j \in \omega$, $i \neq j$: $\langle a_i, b_i \rangle \in \mathrm{Tn}$ *and* $\langle a_i, b_j \rangle \notin \mathrm{Tn}$.

PROOF. (a)–(e) are trivial.

Proof of (f). Clearly (2) $\Rightarrow$ (1). (1) $\Rightarrow$ (2): Assume (1). If ARB/Nb is infinite, let $\{a_i \mid i \in \omega\}$ be a set of representatives from distinct equivalence classes of ARB/Nb; let $b_i = a_i - \bigcup_{j<i} a_j$, then $\{b_i \mid i \in \omega\}$ is as desired. If ARB/Nb is finite, let $\{a_1, \cdots, a_n\} \subseteq \mathrm{ARB} \cap B^{\mathrm{AT}}$ be a set such that for every $D \in (\mathrm{ARB} \cap B^{\mathrm{AT}})/\mathrm{Nb}$: $|D \cap \{a_1, \cdots, a_n\}| = 1$. Then, since RB/Nb is infinite, there is $c \in \mathrm{RB}$ such that $c \subseteq 1^{B^{\mathrm{AT}}} - \bigcup_{i=1} a_i$. Certainly, there is no $d \subseteq c$ such that $d \in \mathrm{ARB}$. So again it is easy to construct an infinite subset of RB of pairwise disjoint elements.

(3) $\Rightarrow$ (1): Assume $\neg$ (1). For $f, g \in \mathrm{Aut}(B)$ we define $f \sim g$ iff for every $a \in \mathrm{RB}$ $\langle f(a), g(a) \rangle \in \mathrm{Nb}$. Clearly $\sim$ is an equivalence relation with finitely many equivalence classes, and $f \sim g$ iff $fg^{-1} \in \mathrm{Aut}_0(B)$. So $\mathrm{Aut}(B)/\mathrm{Aut}_0(B)$ is finite and $\neg$ (3) is proved.

(1) $\Rightarrow$ (3): Distinguish again between the cases that ARB/Nb is finite, or infinite; in both cases the proof is very easy.

(g) Let $\{a_i \mid i \in \omega\}$, $\{b_i \mid i \in \omega\}$ be sets of pairwise disjoint elements of $\mathrm{RB} \cap B^{\mathrm{AT}}$, $\mathrm{RB} \cap B^{\mathrm{AL}}$ respectively, such that for every $i \in \omega$, $a_i \cup b_i \in B$. We first show: (∗) If $i \neq j$ and $\langle a_i, b_j \rangle \in \mathrm{Tn}$, then for every $f \in \mathrm{Aut}(B)$: if $f(a_i) \cap a_i \in \mathrm{RS}$, then $\langle f(a_i), a_j \rangle \in \mathrm{Nb}$. So suppose $i \neq j$, $\langle a_i, b_j \rangle \in \mathrm{Tn}$, and let $f \in \mathrm{Aut}(B)$ and $f(a_i) \cap a_i \in \mathrm{RS}$.

There are $a'_i \subseteq a_i$, $b'_i \subseteq b_i$ such that $\langle a'_i, a_i \rangle$, $\langle b'_i, b_i \rangle \in Nb$ and $f(a'_i \cup b'_i) \cap (a'_i \cup b'_i) = 0$. Clearly $a'_i \cup b'_i \in B$, and by (c) and (d) w.l.o.g. $a_i = a'_i$, $b_i = b'_i$, so $f(a_i \cup b_i) \cap (a_i \cup b_i) = 0$. Suppose by contradiction $\langle a_j, f(a_i) \rangle \notin NB$. If $a_j - f(a_i) \notin RS$, then since $a_i \cup b_i$, $a_j \cup b_j \in B$, $b_j - f(b_i) \notin RS$.

Let $g = f \restriction (a_i \cup b_i \cup f(a_i \cup b_i)) \cup \mathrm{Id} \restriction (1 - (a_i \cup b_i \cup f(a_i \cup b_i)))$; then $g(a_i) \cap$

$a_i = 0$ but $g(b_j) \cap b_j = b_j - f(b_i) \not\in RS$, a contradiction. A similar contradiction is obtained if we assume that $a_j - f(a_i) \in RS$ but $f(a_i) - a_j \not\in RS$. So (*) is proven.

(*) implies that for every $i \in \omega \,|\{j \mid \langle a_i, b_j \rangle \in \mathrm{Tn}\}| \leqq 2$. Since for every $i \in \omega$, $a_i \cup b_i \in B$, for every $f \in \mathrm{Aut}(B)$: $f(a_i) \cap a_i \in \mathrm{RS}$ iff $f(b_i) \cap b_i \in \mathrm{RS}$. This implies that for every $i, j \in \omega$: $\langle a_i, b_j \rangle \in \mathrm{Tn}$ iff $\langle a_j, b_i \rangle \in \mathrm{Tn}$. So, the relation $\{\langle i, j \rangle \mid \langle a_i, b_j \rangle \in \mathrm{Tn}\}$ is an equivalence relation, and all of its equivalence classes are of power $\leqq 2$, so (g) follows.

Let $K_0^1 = \{B \mid B \in K_0$ and $\mathrm{Aut}(B)/\mathrm{Aut}_0(B)$ is finite$\}$, $K_0^2 = K_0 - K_0^1$ and $K_i^j = \{M_i(B) \mid B \in K_0^j\}$. So there is a sentence φ such that for every $B \in K_0$, $\mathrm{Aut}(B) \models \varphi$ iff $B \in K_0^1$.

Lemma 3.3. *Let* $M_{13}(B) = \langle \mathrm{Aut}(B), (B^{\mathrm{TL}})^{\mathrm{II}}; \mathrm{Op} \rangle$ *and* $K_{13}^2 = \{M_{13}(B) \mid B \in K_0^2\}$ *then* K_{13}^2 *is explicitly interpretable in* K_{12}^2.

Proof. Let $\varphi_8(C, D)$ be the formula in the language of K_{12} which says $C \subseteq B^{\mathrm{AT}}$, $D \subseteq B^{\mathrm{AL}}$ and there exist $C_1 \subseteq B^{\mathrm{AT}}$, $D_1 \subseteq B^{\mathrm{AL}}$ such that $|C| = |C_1|$, $|D| = |D_1|$, the relation $\{\langle c, d \rangle \mid c \in C_1,\ d \in D_1$ and $\langle c, d \rangle \in \mathrm{Tn}\}$ is a 1–1 correspondence between C_1 and D_1. Then obviously if $B \in K_0^2$, $C, D \in |M_{12}(B)|$ then: $M_{12}(B) \models \varphi_8[C, D]$ iff $C \subseteq B^{\mathrm{AT}}$, $D \subseteq B^{\mathrm{AL}}$ and $|C| = |D|$. We now show how to interpret in $M_{12}(B)$, $B \in K_0^2$ equivalence relations of $B^{\mathrm{AT}} \cup B^{\mathrm{AL}}$. Let $g: B^{\mathrm{AT}} \to B^{\mathrm{AT}}$, $h: B^{\mathrm{AL}} \to B^{\mathrm{AL}}$ be arbitrary functions and let

$$E_{\langle g,h \rangle} = \{\langle x, y \rangle \mid x, y \in B^{\mathrm{AL}} \cup B^{\mathrm{AT}} \quad \text{and} \quad |g^{-1}(\{x\}) \cup h^{-1}(\{x\})| = |g^{-1}(\{y\}) \cup h^{-1}(\{y\})|\}.$$

The functions from B^{AL} to B^{AL} and from B^{AT} to B^{AT} belong to $M_{12}(B)$; when $B \in K_0^2$ $E_{\langle g,h \rangle}$ is definable from $\langle g, h \rangle$ in $M_{12}(B)$; and when $\langle g, h \rangle$ ranges over all possible pairs, $E_{\langle g,h \rangle}$ ranges over all equivalence relations on $B^{\mathrm{AT}} \cup B^{\mathrm{AL}}$. $(B^{\mathrm{TL}})^{\mathrm{II}}$ can easily be interpreted in $(B^{\mathrm{AT}} \cup B^{\mathrm{AL}})^{\mathrm{II}}$. So the lemma is proved.

Let $M_{14}(B) = \langle B^{\mathrm{TL}}, \mathrm{Aut}^f(B); \mathrm{Op}, \cdots \rangle^{\mathrm{II}}$ where

$$\mathrm{Aut}^f(B) = \{g \restriction D \mid g \in \mathrm{Aut}(B), D \subseteq B^{\mathrm{TL}} \quad \text{and} \quad |D| < \aleph_0\}$$

and

$$K_{14}^2 = \{M_{14}(B) \mid B \in K_0^2\}.$$

Certainly K_{14}^2 is explicitly interpretable in K_{13}^2.

Lemma 3.4. $(V = L)$ *Let* $A \in K_0^2$, B *be any BA such that* $|B| \leqq \aleph_0$ *and* $\mathrm{Aut}(A) \equiv \mathrm{Aut}(B)$, *then* $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$.

PROOF. There is a sentence χ in the language of groups such that for every BA C if $|C| \leqq \aleph_0$, then $\mathrm{Aut}(C) \models \chi$ iff $C \in K_0^2$. So since $\mathrm{Aut}(A) \equiv \mathrm{Aut}(B)$ and $A \in K_0^2$, $B \in K_0^2$. So $M_{14}(A) \equiv M_{14}(B)$. By 2.19 $\langle A^{\mathrm{TL}}, \mathrm{Aut}^f(A); \cdots \rangle \cong \langle B^{\mathrm{TL}}, \mathrm{Aut}^f(B); \cdots \rangle$. So $M_{14}(A) \cong M_{14}(B)$. Let $\varphi_9(f)$ be the formula in the language of K_{14}^2 which says: $f \in \mathrm{Aut}(B^{\mathrm{TL}})$, and for every finite subset $D \subseteq B^{\mathrm{TL}}$: $f \restriction D \in \mathrm{Aut}^f(B)$. It is easily seen that for every $C \in K_0^2$: $\mathrm{Aut}(C) = \{f \mid M_{14}(C) \models \varphi_9[f]\}$. So since for every $C \in K_0^2$: $\mathrm{Aut}(C)$ is defined in $M_{14}(C)$ by the formula φ_9, and $M_{14}(A) \cong M_{14}(B)$, $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$. So the lemma is proved.

Let $\mathrm{Aut}_1(B)$ be the following subgroup of $\mathrm{Aut}(B^{\mathrm{TL}})$:

$$\mathrm{Aut}_1(B) = \{f \mid f \in \mathrm{Aut}(B^{\mathrm{TL}}) \text{ and for some } f_1, f_2 \in \mathrm{Aut}(B):$$

$$f \restriction B^{\mathrm{AT}} = f_1 \restriction B^{\mathrm{AT}} \text{ and } f \restriction B^{\mathrm{AL}} = f_2 \restriction B^{\mathrm{AL}}\}.$$

LEMMA 3.5. (a) *If* $a \in \mathrm{ARB}(B)$ *and* $|\{f(a) \mid f \in \mathrm{Aut}(B)\}/\mathrm{Nb}| > 2$, *then for every* $b \in B^{\mathrm{AT}} \cup B^{\mathrm{AL}}$: $\langle a, b \rangle \in \mathrm{Tn}$ *or* $\langle b, a \rangle \in \mathrm{Tn}$ *iff* $a \cup b \in B$.

(b) *If* $a \in \mathrm{ARB}(B)$ *and* $|\{f(a) \mid f \in \mathrm{Aut}(B)\}/\mathrm{Nb}| = 2$, *then for every* $b \in B^{\mathrm{AT}} \cup B^{\mathrm{AL}}$: $\langle a, b \rangle \in \mathrm{Tn}$ *or* $\langle b, a \rangle \in \mathrm{Tn}$ *iff either* $a \cup b \in B$, *or for every* $f \in \mathrm{Aut}(B)$: *if* $f(a) \cap a \in \mathrm{RS}$, *then* $f(a) \cup b \in B$.

(c) *Let* $B \in K_0^1$, $|(\mathrm{ARB} \cap B^{\mathrm{AT}})/\mathrm{Nb}| = k$, $\{a_1, \cdots, a_k\} \subseteq \mathrm{ARB} \cap B^{\mathrm{AT}}$ *and if* $i \neq j$ $\langle a_i, a_j \rangle \notin \mathrm{Nb}$, *and let* $b_1, \cdots, b_k$ *be such that for every* $1 \leqq i \leqq k$: $\langle a_i, b_i \rangle \in \mathrm{Tn}$. *Then*: (1) *for every* $f \in \mathrm{Aut}_1(B)$, *there are permutations* $\tilde{f}^{\mathrm{AT}}$, $\tilde{f}^{\mathrm{AL}}$ *of* $\{1, \cdots, k\}$, *such that for every* $1 \leqq i \leqq k$ $\langle f(a_i), a_{\tilde{f}^{\mathrm{AT}}(i)} \rangle \in \mathrm{Nb}$, *and* $\langle f(b_i), b_{\tilde{f}^{\mathrm{AL}}(i)} \rangle \in \mathrm{Nb}$; (2) *for every* $f \in \mathrm{Aut}_1(B)$: $f \in \mathrm{Aut}_0(B)$ *iff* $\tilde{f}^{\mathrm{AT}} = \tilde{f}^{\mathrm{AL}}$.

PROOF. (a), (b) and (c) (1) are trivial. It is easy to see that if $f \in \mathrm{Aut}(B)$, then $\tilde{f}^{\mathrm{AT}} = \tilde{f}^{\mathrm{AL}}$. We now prove that if $f \in \mathrm{Aut}_1(B)$ and $\tilde{f}^{\mathrm{AT}} = \tilde{f}^{\mathrm{AL}}$, then $f \in \mathrm{Aut}(B)$.

By (a) and (b) we can w.l.o.g. assume that for some $r \geqq 0$: for every $i \leqq r$: $a_{2i-1} \cup b_{2i}$, $a_{2i} \cup b_{2i-1} \in B$, and for every $2r < i \leqq k$: $a_i \cup b_i \in B$. Let a_0 be the maximal atomic element of B^{TL}, $a_{k+1} = a_0 - \bigcup_{i=1}^k a_i$ and $b_{k+1} = -a_0 - \bigcup_{i=1}^k b_i$ then $a_{k+1} \cup b_{k+1} \in B$, and it is easy to see that for every $g \in \mathrm{Aut}_1(B)$, $a \subseteq a_{k+1}$ and $b \subseteq b_{k+1}$: $(a - g(a)) \cup (g(a) - a) \in \mathrm{As}(B)$ and $(b - g(b)) \cup (g(b) - b) \in \mathrm{Al}(B) \cup \{0\}$.

In order that $f \in \mathrm{Aut}(B)$, it suffices that for every $c \in B$: $f(c) \in B$; so let $c \in B$. For every $1 \leqq i \leqq k+1$ let $c_i = c \cap (a_i \cup b_{i'})$, where $i' = i$ for $2r < i \leqq k+1$, $(2i-1)' = 2i$ and $(2i)' = 2i - 1$ for $1 \leqq i \leqq r$; $c_i \in B$, so it is sufficient to show that for every $1 \leqq i \leqq k+1$: $f(c_i) \in B$. Certainly $(c_{k+1} - f(c_{k+1})) \cup (f(c_{k+1}) - c_{k+1}) \in I(B)$, so $f(c_{k+1}) \in B$. Suppose $2r < i \leqq k$. If $c_i \cap a_i \in \mathrm{As}(B)$, then $c_i \in I(B)$, so $f(c_i) \in I(B) \subseteq B$. Otherwise $\langle c_i \cap a_i, a_i \rangle$, $\langle c_i \cap b_i, b_i \rangle \in \mathrm{Nb}$, so

$\langle f(a_i \cap c_i), a_{\bar f^{AT}(i)}\rangle$, $\langle f(b_i \cap c_i), b_{\bar f^{AT}(i)}\rangle \in \mathrm{Nb}$. Since $\bar f^{AT}(i) > 2r$, $a_{\bar f^{AT}(i)} \cup b_{\bar f^{AT}(i)} \in B$, so $f(c_i) = f(a_i \cap c_i) \cup f(b_i \cap c_i) \in B$. A similar argument holds when $i \leqq 2r$. So $f(c) \in B$, and the lemma is proved.

LEMMA 3.6. *($V = L$) Let $A \in K^1_0$, B be any BA such that $|B| \leqq \aleph_0$ and* $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$, *then* $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$.

PROOF. Suppose Nb has exactly $2n$ equivalence classes in A. Let $P_1, \cdots, P_n$ be the set of all (distinct) equivalence classes of Nb which are included in A^{AT} and $Q_1, \cdots, Q_n$ be those which are included in A^{AL}; we choose the Q_i's so that for every i, $a \in P_i$ and $b \in Q_i$, implies $\langle a, b\rangle \in \mathrm{Tn}$. Let $\vec P = \langle P_1, \cdots, P_n\rangle$, $\vec Q = \langle Q_1, \cdots, Q_n\rangle$ and $M_{15}(A, \vec P, \vec Q) = \langle \mathrm{Aut}(A), \langle A^{AT}; \vec P\rangle^{II}, \langle A^{AL}; \mathrm{Al}(B), \vec Q\rangle^{II}; \mathrm{Op}\rangle$ where the P_i's and Q_i's are considered to be unary predicates. Let T be the complete theory of $M_{15}(A, \vec P, \vec Q)$ and $K^T_{15} = \{M_{15}(A', \vec P', \vec Q') \mid M_{15}(A', \vec P', \vec Q') \models T\}$. It is easily seen that there are $P'_i, Q'_i \subseteq B$ such that $M_{15}(B, \vec P', \vec Q') \in K^T_{15}$. Under the assumption that $V = L$ we shall show that all the models of K^T_{15} are isomorphic. Let C be a BA and $\mathrm{Aut}^{ATf}(C) = \{f \restriction D \mid f \in \mathrm{Aut}\ (B) \text{ and } D \text{ is a finite subset of } C^{AT}\}$; Aut^{ALf} is defined similarly.

Suppose $M_{15}(C, \vec P, \vec Q) \in K^T_{15}$ and let

$$M_{16}(C, \vec P, \vec Q) = \langle\langle \mathrm{Aut}^{ATf}(C), C^{AT}, \vec P, \mathrm{Op}\rangle^{II}, \langle \mathrm{Aut}^{ALf}(C), C^{AL}; \vec Q, \mathrm{Op}\rangle^{II}\rangle$$

and $K^T_{16} = \{M_{16}(C, \vec P, \vec Q) \mid M_{15}(C, \vec P, \vec Q) \models T\}$.

Certainly K^T_{16} is explicitly interpretable in K^T_{15}, and we shall show that also K^T_{15} is explicitly interpretable in K^T_{16}.

Let $\mathrm{Aut}^{AT}(C) = \{f \restriction C^{AT} \mid f \in \mathrm{Aut}(C)\}$ and $\mathrm{Aut}^{AL}(C)$ be defined similarly. We first show that $f \in \mathrm{Aut}^{AT}(C)$ iff $f \in \mathrm{Aut}(C^{AT})$ and for every finite $D \subseteq C^{AT}$ $f \restriction D \in \mathrm{Aut}^{ATf}(C)$. For let $a_1, \cdots, a_n, b_1, \cdots, b_n$ belong to $P_1, \cdots, P_n, Q_1, \cdots, Q_n$ respectively so $f \restriction \{a_1, \cdots, a_n\}$ can be extended to an automorphism g of C. It is easy to see that $f \cup (g \restriction B^{AL})$ can be extended uniquely to an automorphism of C. A similar statement holds for C^{AL}.

Suppose $M_{16}(C, \vec P, \vec Q) \in K^T_{16}$ and let

$$M_{17}(C, \vec P, \vec Q) = \langle\langle \mathrm{Aut}^{AT}(C), C^{AT}; \vec P, \mathrm{Op}\rangle, \langle \mathrm{Aut}^{AL}(C), C^{AL}; \vec Q, \mathrm{Op}\rangle\rangle,$$

and and $K^T_{17} = \{M_{17}(C, \vec P, \vec Q) \mid M_{16}(C, \vec P, \vec Q) \in K^T_{16}\}$. It follows that K^T_{17} is explicitly interpretable in K^T_{16}. From 3.6(c) it follows that K^T_{15} is explicitly interpretable in K^T_{17}, so we have proved that K^T_{15} is explicitly interpretable in K^T_{16}.

By 2.19 all the models of K^T_{16} are isomorphic, so the same is true for K^T_{15}. So $M_{15}(B, \vec P', \vec Q') \cong M_{15}(A, \vec P, \vec Q)$ so $\mathrm{Aut}(B) \cong \mathrm{Aut}(A)$. Q.E.D.

Corollary 3.7. *($V = L$) If A and B are any countable or finite BA's and* $\mathrm{Aut}(A) \equiv \mathrm{Aut}(B)$ *then* $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$.

Proof. If $A \in K_0$ then the claim of the corollary follows from 3.5 and 3.6; if $A \notin K_0$ then it follows easily from 2.1.

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